Modular Invariance as an Explanation for the Absence of Monopoles

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Abstract

It is shown that modular invariance provides a natural explanation for the absence of monopoles when assumed to be a discrete gauge symmetry (i.e. the physical states are identified with the orbits along the modular group).

It follows that monopoles can not be seen because it is always possible to find a suitable gauge-fixing in which they are not present. This result relies upon an easy to prove but non-trivial property of the modular group (a sort of no-monopole theorem for a diophantine equation).

A modular-invariant formulation for the hamiltonian of the electromagnetism is given. No monopole arises if independent modular transformations are allowed at each point in space-time where point-like charges are present.

Modular invariance is the quantum analogue of the classical property that equivalent equations of motion for the electric and magnetic fields are obtained from any quadratic form hamiltonian density $H = a\vec{E}^2 + b\vec{E} \cdot \vec{B} + c\vec{B}^2$, with a, b, c real, a, c > 0 and $\Delta = b^2 - 4ac < 0$; it is therefore quite natural to regard modular invariance as a gauge symmetry.

1 Introduction.

The modular group $PSL(2, \mathbf{Z})$ has received recently a lot of attention in connection with the problem of the electro-magnetic duality (see e.g. [1]-[5]).

In this paper I wish to point out that the modular invariance provides a natural explanation for both the integral quantization of the electric charge and the experimental non-observation of the magnetic monopoles. If modular invariance is regarded as a discrete gauge symmetry (which means that the physical states in the Hilbert space are identified with the orbits along the modular group), then the two properties above follow immediately. This result is a consequence of an easy-to-prove but highly non-trivial property of the modular group (let me call this simple theorem concerning a diophantine equation the no-monopole theorem; it is demonstrated in the appendix). Its statement is the following: any two-component vector $\begin{pmatrix} n \\ m \end{pmatrix}$, with n,m integers (n can be assumed to represent the electric charge, m the magnetic charge), can be carried with $PSL(2, \mathbf{Z})$ -modular transformations to a vector of the kind $\begin{pmatrix} p \\ 0 \end{pmatrix}$ where p is a uniquely determined integer which labels the set of inequivalent orbits of the modular group.

From this result follows that it is always possible to find a suitable gauge-fixing in which the monopoles (having non-vanishing magnetic charge) are not present. The dual nature of the electric and magnetic field allows defining the electric field as the one which, in the given gauge-fixing, is associated with the charged matter; conversely the magnetic field is the one associated with uncharged matter.

The above-cited no-monopole theorem can be regarded just as a funny property of the modular group with no real physical implications if the modular invariance is not a symmetry of the dynamics. However it can be easily provided a modular-invariant formulation for the hamiltonian of the electro-magnetism. The absence of monopoles in this case is guaranteed if we allow making independent modular transformations at each point of the space-time where point-like charges are present¹.

In this framework it is more convenient to work with the physical electric and magnetic fields \vec{E}, \vec{B} instead of the unphysical gauge-potential A_{μ} . Manifest Lorentz-covariance is obviously lost but it is recovered at the level of the equations of motion.

The hamiltonian H can be defined to be

$$H = \frac{1}{2} \int d^3 \vec{x} (a\vec{E}^2 + b\vec{E} \cdot \vec{B} + c\vec{B}^2)$$
 (1)

(couplings with integral-quantized external sources are also allowed), where a, b, c, are all assumed to be integers to take into account the Dirac's integral quantization of the electric and magnetic charges. Moreover a and c are assumed positive and the discriminant $\Delta = b^2 - 4ac < 0$ negative to guarantee the positiveness of the energy ouside the origin.

It is possible to compensate the modular transformations of the electric and magnetic fields \vec{E} , \vec{B} with corresponding modular transformations acting on a,b,c in such a way that the hamiltonian H is formally modular-invariant. In presence of an a priori electrically and/or magnetically charged matter, the no-monopole theorem guarantees that it is always possible to find a gauge-fixing where the magnetic charge is killed.

¹In the following discussion Dirac's type monopoles only are considered.

The fact that modular invariance should be understood as a gauge symmetry and not as a physical one, follows from this argument: at the classical level, where no quantization of the charge is required, the hamiltonian quadratic form (1) can be assumed depending on (this time) real coefficients a, b, c satisfying the same properties as above. The equations of motion derived from any such hamiltonian are equivalent to the equations of motion for the standard hamiltonian with a = c = 1, b = 0 due to linear canonical transformations redefining the fields \vec{E}, \vec{B} . It is therefore natural to regard the modular invariance of the quantized theory not as a physical invariance acting on different physical states, but as a gauge symmetry connecting different formulations of the same theory (a brief remark, advocating the possibility of considering $SL(2, \mathbf{Z})$ as a discrete gauge group can also be found in [3]).

The plan of the paper is the following: in the next section the properties of the modular group are recalled. It will be commented why $PSL(2, \mathbf{Z})$ should be understood as a gauge group and not $SL(2, \mathbf{Z})$. The "no-monopole theorem" will be stated and a modular-invariant quantum toy-model will be discussed. In the following section the above results will be applied to the electromagnetic theory. The proof of the no-monopole theorem is given in the appendix.

2 The modular group and a modular-invariant toymodel.

The modular group $PSL(2, \mathbf{Z})$ is defined as the quotient group $PSL(2, \mathbf{Z}) = SL(2, \mathbf{Z})/\mathbf{Z}_2$, where $SL(2, \mathbf{Z})$ is the group of 2×2 integer-valued matrices with unit determinant:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) \tag{2}$$

with $a, b, c, d \in \mathbf{Z}$ and ad - bc = 1.

 \mathbf{Z}_2 is the subgroup generated by $\pm \mathbf{1}_2$.

 $SL(2, \mathbf{Z})$ admits two generators which are commonly denoted as S, T:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
(3)

In mathematics $PSL(2, \mathbf{Z})$ has a deep geometrical meaning because it characterizes the inequivalent classes of parabolic (genus 1) Riemann surfaces as $H/PSL(2, \mathbf{Z})$, where H is the Poincaré upper half plane (see e.g. the references in [6]).

In physics, besides the applications to string theory, it is now quite popular in connection with the Olive-Montonen electric-magnetic duality realized in four-dimensional supersymmetric theories [5].

In this section I will discuss some properties of the modular group and I will use them to define a modular-invariant toy model which can help clarifying some features shared by more realistic and phenomenologically interesting theories.

Before doing that, let me just make a short comment on the fact that, whenever we have a group G of symmetries it must be left to a final experimental answer the decision whether G acts as a physical group of symmetries, or if some G' subgroup of it (which in some cases can even coincide with G itself) must be considered as a gauge group.

To be more precise, let us suppose starting with a vector space \mathcal{V} which carries a representation of G as a left-action, and let us dispose of a complete set \mathcal{O} of observable operators². Let us suppose moreover that G, due to its adjoint action on \mathcal{O} , acts as automorphism group for \mathcal{O} . In general many possible Hilbert space constructions can be performed out of this framework: if we denote as $|e_n\rangle$ the elements of a basis for \mathcal{V} , a scalar product can be introduced for \mathcal{V} and its dual $\tilde{\mathcal{V}}$, by defining $\langle e_m|e_n\rangle = \delta_{mn}$. In this case \mathcal{V} is a Hilbert space and G acts as group of physical symmetries, i.e. it connects different physical states. But we can also identify the Hilbert space \mathcal{H} as the coset space \mathcal{V}/G' of orbits of \mathcal{V} along the subgroup G'; the scalar product between elements $|p\rangle \in \mathcal{H}$ and $\langle p'| \in \tilde{\mathcal{H}}$ being defined now through

$$\begin{cases} < p'|p>=1, & if |p>, |p'> belong to the same G' orbit, \\ < p'|p>=0, & otherwise. \end{cases}$$

In this case G' is a gauge group. If it is a normal subgroup of G, then G/G' is a factor group which acts as group of physical symmetries for \mathcal{H} .

It is very well known from the experience we have with the electrodynamics that physical consequences on \mathcal{H} can be drawn from the existence of a gauge symmetry, even when G' coincides with G. In this section I will discuss what happens in such a case, when G = G' is the modular group.

In this case as \mathcal{V} we can take the space spanned by the vectors

$$\begin{pmatrix} n \\ m \end{pmatrix}$$
 with n, m integers. (4)

The upper element n can be called the electric charge, and the lower element m the magnetic charge. Under the S, T modular transformations (3) we get respectively (from now on, for simplicity, I will denote with a prime the S-transformed quantities, with a tilde the T-transformed ones):

$$\begin{pmatrix} n \\ m \end{pmatrix} \mapsto \begin{pmatrix} n' \\ m' \end{pmatrix} = \begin{pmatrix} m \\ -n \end{pmatrix}$$

$$\begin{pmatrix} n \\ m \end{pmatrix} \mapsto \begin{pmatrix} \tilde{n} \\ \tilde{m} \end{pmatrix} = \begin{pmatrix} n+m \\ m \end{pmatrix}$$
(5)

A complete set of commuting operators is given by N_e and N_m , respectively the electric and magnetic charge number operators, defined through:

$$N_e \left(\begin{array}{c} n \\ m \end{array} \right) = n \left(\begin{array}{c} n \\ m \end{array} \right)$$

 $^{^{2}}$ I use this word even if at this stage no Hilbert space has been introduced yet; the meaning is simply the following: all the eigenvalues for such operators are real, and there exists a subset of mutually commuting operators which allows to uniquely determine any vector in \mathcal{V} .

$$N_m \left(\begin{array}{c} n \\ m \end{array}\right) = m \left(\begin{array}{c} n \\ m \end{array}\right) \tag{6}$$

 N_e, N_m transform covariantly under modular transformations, as it can be easily checked:

$$\begin{pmatrix} N_e \\ N_m \end{pmatrix} \mapsto \begin{pmatrix} N'_e \\ N'_m \end{pmatrix} = S^{-1} \begin{pmatrix} N_e \\ N_m \end{pmatrix} = \begin{pmatrix} -N_m \\ N_e \end{pmatrix}$$

$$\begin{pmatrix} N_e \\ N_m \end{pmatrix} \mapsto \begin{pmatrix} N_e^{\tilde{}} \\ N_m^{\tilde{}} \end{pmatrix} = T^{-1} \begin{pmatrix} N_e \\ N_m \end{pmatrix} = \begin{pmatrix} N_e - N_m \\ N_m \end{pmatrix}$$
(7)

A fundamental property of the $PSL(2, \mathbf{Z})$ modular group has already been stated in the introduction: it is the content of what can be called "the no-monopole theorem": each $\begin{pmatrix} n \\ m \end{pmatrix}$ vector lies in the $PSL(2, \mathbf{Z})$ -orbit of a vector of the form $\begin{pmatrix} p \\ 0 \end{pmatrix}$, where p is an integer and it is uniquely defined (no other vector $\begin{pmatrix} \hat{p} \\ 0 \end{pmatrix}$ with the integer $\hat{p} \neq p$ lies in

the same $PSL(2, \mathbf{Z})$ -orbit). Therefore the whole set of inequivalent $PSL(2, \mathbf{Z})$ -orbits is labelled by an integer, given by p. The proof of this theorem should be found somewhere in the mathematical literature, however it is so simple that it is more convenient to show it directly: it is given in the appendix.

Therefore if we regard the set of states connected by $PSL(2, \mathbf{Z})$ -transformations not as distinct physical states, but as different gauge-choices of a unique physical state, then we can always find a suitable gauge-fixing which shows the absence of magnetic charged matter.

Obviously we are free to choose any other gauge fixing (such as $\begin{pmatrix} 0 \\ -p \end{pmatrix}$, which is related to the first one by an S-transformation), but in that case we are also free to redefine our electric and magnetic charge number operators in such a way that the N_m operator always admits zero-eigenvalues only.

Notice that, even if from a mathematical point of view we can identify the set of orbits with the integer numbers p (and the N_m operator is "needless", so to speak, because of its triviality), on physical grounds things are not precisely like that: both the N_e and N_m operators are (covariantly transforming) physical observables: indeed monopoles can in principle be actually detected with an experiment (and up to now N_m has been proven to have only zero-eigenvalue, apart perhaps the single event found by Cabrera).

It is interesting to notice that we could have not considered the full $SL(2, \mathbf{Z})$ group as a gauge group. This bigger group admits also the change in sign, and for that reason $\begin{pmatrix} \pm p \\ 0 \end{pmatrix}$ belong to the same $SL(2, \mathbf{Z})$ -orbit. Therefore in a world in which $SL(2, \mathbf{Z})$ acts as a gauge group, we have no way to distinguish between positive and negative values of the electric charge: only the absolute value of the charge matters. This unphysical property obliges ourselves to restrict our attention to $PSL(2, \mathbf{Z})$.

There is a priori no reason why $PSL(2, \mathbf{Z})$ should act as a gauge group, just as there is no reason why it should act as a physical group of symmetry. Both possibilities are logically consistent. Obviously it can happen that only a subgroup of $PSL(2, \mathbf{Z})$ should

be considered as a gauge group (for instance the discrete subgroup $\Gamma(2)$, see [6]); in this case restrictions on the allowed spectrum of monopoles can be drawn.

The construction which leads us to the "no-monopole theorem" allows us also to introduce a modular-invariant P operator. It is simply defined through the position:

$$P\left(\begin{array}{c} n\\m \end{array}\right) = p(n,m)\left(\begin{array}{c} n\\m \end{array}\right) \tag{8}$$

where p(n, m) is the integer labelling the orbit of $\binom{n}{m}$ as specified above. The operator P is therefore modular-invariant by construction.

Any function of P which has the right property of being bounded below, etc., can be considered as a modular-invariant hamiltonian, providing a modular-invariant dynamics. The simplest, and in some sense most natural choice for such operators, is to take for hamiltonian

$$H = P^2 (9)$$

When restricted the Hilbert space to the space of $PSL(2, \mathbf{Z})$ -orbits³ we get that, apart the vacuum state, any other energy state is doubly degenerated, corresponding to the charge-invariance $(P \mapsto -P)$ of such a hamiltonian. The spectrum of the theory is given by the set of squared integers p^2 (and coincides, apart the degeneracy, with the spectrum of the infinite-potential double well).

In this section I have provided the logical ground for the claim that modular invariance can "kill" the monopoles. At first sight it would seem unjustified to identify the upper and lower number operators acting on the vectors in \mathcal{V} with, respectively, the electric and magnetic charge number operators. In the next section I will show that the hamiltonian of the electrodynamics can be naturally formulated in a modular-invariant framework and such an identification will no longer appear so arbitrary.

3 Modular invariance of the electromagnetic hamiltonian.

In this section I will furnish a dynamical content to the previous discussion by showing that the hamiltonian of the electromagnetism naturally carries a modular invariance when the integral quantization of the electric and magnetic charges are taken into account⁴.

Before introducing the quantum case, I will discuss the classical one.

Let me at first mention that I will call the Maxwell equations put in standard form the set of Maxwell equations for the electric and magnetic fields \vec{E}, \vec{B} with the normalizations

³ then P coincides with N_e in the gauge-fixing specified above.

⁴Due to Dirac's result, a quantum theory necessarily involves integral-quantized electric and magnetic charges (multiples of e, b respectively). Conversely an integral charge-quantized theory need not be a quantum one. In any case I will deserve the name classical for theories not admitting charge-quantization.

given as follows:

$$\dot{\vec{E}} = \vec{\nabla} \times \vec{B} + \vec{J}
\dot{\vec{B}} = -\vec{\nabla} \times \vec{E} + \vec{K}
\vec{\nabla} \cdot \vec{E} = -J^0
\vec{\nabla} \cdot \vec{B} = -K^0$$
(10)

 $(J^{\mu}=J^{0},\vec{J})$ and $(K^{\mu}=K^{0},\vec{K})$ are respectively the electric and magnetic quadricurrents satisfying the continuity equations

$$\dot{J}^0 + \vec{\nabla} \cdot \vec{J} = 0 \tag{11}$$

(and similarly for (K^0, \vec{K})).

The absence of monopoles requires that it is always possible to set the magnetic quadricurrent equal to zero $(K^0 = \vec{K} = 0)$.

In the classical case, in absence of matter, the free hamiltonian which allows deriving the Maxwell equations above with $J^{\mu} = K^{\mu} = 0$ is given by

$$H = \frac{1}{2} \int d^3 \vec{x} (\vec{E}^2 + \vec{B}^2) \tag{12}$$

where the following Poisson brackets are assumed:

$$[E_j(\vec{x},t), B_k(\vec{y},t)] = i\epsilon_{jkl} \frac{\partial}{\partial n^l} \delta(\vec{x} - \vec{y})$$
(13)

where j, k, l = 1, 2, 3 are spatial indices and ϵ_{jkl} is the totally antisymmetric tensor

$$(\epsilon_{123}=1).$$

All the other Poisson brackets are assumed vanishing.

The group of invariances of the above hamiltonian is rather poor: it is just given by the parity transformation:

$$\vec{E} \mapsto \vec{E},$$
 $\vec{B} \mapsto -\vec{B}$ (14)

and the S-duality

$$\begin{array}{ccc} \vec{E} & \mapsto & \vec{B}, \\ \vec{B} & \mapsto & -\vec{E}. \end{array}$$

However it should be noticed that the above hamiltonian is not the most general one satisfying the positivity condition for the energy (outside the zero-energy solution $\vec{E} = \vec{B} = 0$).

Indeed any hamiltonian such as

$$\hat{H} = \frac{1}{2} \int d^3 \vec{x} (a\vec{E}^2 + b\vec{E} \cdot \vec{B} + c\vec{B}^2)$$
 (15)

with a,b,c real numbers, a,c>0 and negative discriminant $\Delta=b^2-4ac<0$, can be used as well. It should also be remarked however that H in (12) and \hat{H} in (15) do not provide different dynamics. It is indeed true that there exists a canonical transformation for the fields \vec{E}, \vec{B} , such as

$$\vec{E}^{ct} = \sqrt{a}\vec{E} + \frac{b}{2\sqrt{a}}\vec{B}$$

$$\vec{B}^{ct} = \frac{1}{\sqrt{a}}\vec{B}$$
(16)

which transforms the equations of motion for the fields \vec{E} , \vec{B} derived from (15) into the canonical Maxwell equations (10) for the fields \vec{E}^{ct} , \vec{B}^{ct} (in absence of matter).

When taking into account the interaction with matter, and assuming the integral quantization of the charges, it is natural to restrict ourselves to hamiltonian of the form (15), where now the coefficients a, b, c are assumed integer-valued, and satisfying of course the same conditions as before:

$$a, c > 0$$

$$\Delta = b^2 - 4ac < 0 \tag{17}$$

In this case \hat{H} easily acquires the structure of a formally modular-invariant operator; indeed we can compensate the modular transformations for the vector $\begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}$:

$$\begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} \mapsto \begin{pmatrix} \vec{E'} \\ \vec{B'} \end{pmatrix} = S \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}$$

$$\begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} \mapsto \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} = T \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}$$
(18)

(where S, T are given in (3) and \vec{E}, \vec{B} transform like n, m in the previous section), with corresponding covariant modular transformations on the coefficients a, b, c, in such a way that \hat{H} is left invariant, as it can be easily checked:

$$a \mapsto a' = c$$

$$b \mapsto b' = -b$$

$$c \mapsto c' = a \tag{19}$$

and

$$a \mapsto a = a$$

 $b \mapsto b = b - 2a$
 $c \mapsto c = a + c - b$ (20)

The above transformations are modular-covariant since a, b, c transform like

$$a \equiv N_m^2; \quad b \equiv 2N_e N_m; \quad c \equiv N_e^2.$$
 (21)

where N_e , N_m are the charge number operators introduced in (6).

Notice that the property (17,1) is mantained by the above modular transformations for a, b, c, while the discriminant Δ in (17,2) is left invariant, which implies the consistency of our assumptions.

In this framework it is quite evident that the hamiltonian of the electrodynamics possesses a modular invariance and that such modular invariance must be understood as a gauge symmetry.

Of course we can use our freedom to choose the overall normalization factor for \hat{H} to set $\Delta = -4$, which corresponds to the canonical choice (12) for the hamiltonian (a = c = 1, b = 0).

Let us discuss now a bit more carefully the equations of motion derived from the free (15) (or equivalently (12)) hamiltonian: it should be noticed that only the first two Maxwell equations (those involving the time derivative) are derived from the Poisson brackets, and the currents are automatically set equal to zero.

The second set of Maxwell equations are not obtained as the previous ones by making use of the Poisson brackets, rather they are derived from the first two equations after taking into account a time-integration. Therefore the electric and magnetic charge densities J^0, K^0 do not appear in the hamiltonian but must be inserted as boundary conditions. The continuity equations for the quadricurrents imply that they are distributions of electric and magnetic charges at rest in the given reference system. The boundary conditions must be specified according to the principles we have in mind. Let us imagine having a distribution of point-like electric and magnetic charges. Taking into account the discussion of the previous section, and assuming that we can perform independent modular transformations at each point in the space (for a fixed time) where $\binom{n_e}{n_m}$ -electric and magnetic charges are present (notice that such modular transformations are indeed invariances of the dynamics, since the charge-distribution is decoupled from the hamiltonian), we can therefore conclude that the electric-magnetic charged matter can be recasted into electric charged matter only.

The assumption that independent modular transformations can be performed is really a strong assumption, but it is very plausible. It looks indeed like a sort of principle of general covariance for the modular transformations or, stated otherwise, the modular group should be seen as a local discrete gauge group.

At this point one can ask what happens when not only charge distributions at rest, but also currents are taken into account. The simplest case involves taking an extra-term

$$m_e \vec{L} \cdot \vec{B} + m_m \vec{L} \cdot \vec{E} \tag{22}$$

in the hamiltonian density (15). m_e, m_m are integers denoting the integral quantization of the electric and magnetic charges respectively.

The above term (22) is modular-invariant provided that m_e, m_m transform under modular transformations like the (N_e, N_m) charge number operators (6), whose transformations are given by (7):

$$(m_e \equiv N_e, m_m \equiv N_m)$$

We can therefore always set the gauge-fixing where only electric charged matter is present $(m_m = 0, m_e = p)$. In such a gauge the coefficients a, b, c will be generic integers satisfying the (17) conditions but, as it can be easily checked, by performing canonical transformations of the \vec{E}, \vec{B} fields, one can derive the standard Maxwell equations (10) in presence of electric current only $(\vec{J} = p\vec{\nabla} \times L)$, which is in this case divergenceless.

If extra currents are present we can make use of the same principle of considering the modular group $PSL(2, \mathbf{Z})$ a discrete "local" gauge group, to put them in the same form as above to obtain the vanishing of the magnetic currents.

4 Appendix: the "no-monopole" theorem.

Let

$$\hat{M} = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$$

be a generic matrix in $SL(2, \mathbf{Z})$.

I will fix uniquely the corresponding element M in $PSL(2, \mathbf{Z})$ by requiring the following convention being satisfied:

M will be taken the element having c > 0 if $c \neq 0$; otherwise the one with d > 0.

Let us prove first that if $\begin{pmatrix} p \\ 0 \end{pmatrix}$ exists in the $PSL(2, \mathbf{Z})$ -orbit, then it is unique; that is easy:

$$M\left(\begin{array}{c}p\\0\end{array}\right) = \left(\begin{array}{c}\hat{p}\\0\end{array}\right)$$

requires c = 0 and, for the condition on the determinant, ad = 1; since a, d are integers, necessarily

$$a = d = 1.$$

Let us show now that the equation

$$M\left(\begin{array}{c}n\\q\end{array}\right) = \left(\begin{array}{c}\hat{n}\\0\end{array}\right)$$

always admits solutions $M \in PSL(2, \mathbf{Z})$.

Without loss of generality we can take q > 0.

If n is a multiple of q: n = tq, then many such matrices M can be found; they are given by

$$\left(\begin{array}{cc} k & -kt - 1 \\ 1 & -t \end{array}\right)$$

for any integer-valued k.

If n is not a multiple of q, let

$$n = tq + r,$$

with $1 \le r \le q - 1$.

Let us now define the relatively prime positive integers x, y through the position

$$\frac{x}{y} = \frac{r}{q}.$$

We must satisfy the two conditions

$$ad - bc = 1$$
$$cn + dq = 0$$

The second condition tells us that c must be a multiple of y:

$$c = ky$$

for some integer k.

Then

$$d = -k(ty + x)$$

and inserting these values in the equation for the determinant we get

$$-k[y(at+b)+ax] = 1.$$

Necessarily k = 1 (remember that c > 0) and we are reduced to find integer solutions to the equation

$$my + m'x = -1$$

with x, y relatively prime positive integers.

The fact that such equation always admits solution is of course one of the basic theorem in number theory (see however [7]).

5 Conclusions.

In this paper I have furnished some evidence of the fact that the modular group $PSL(2, \mathbf{Z})$ can be considered as a discrete local gauge group and that, in such a case, some interesting conclusions can be drawn; in particular the hamiltonian of the electromagnetism it is shown to be modular-invariant and this naturally lead us, if the above interpretation proves to be correct, to the absence of monopoles.

It would be interesting to understand the above result in a completely field-theoretical lagrangian framework. The price we have to pay in this case is the presence of unphysical degrees of freedom, but we could dispose of a manifestly Lorenz-covariant framework. Indeed duality invariant actions for the electromagnetism have been discussed in [2, 8].

I wish also to point out that it is hardly conceivable to find a relation between this framework to explain the absence of monopoles and the celebrated Yang-Wu [9] result concerning their absence in trivial fiber-bundles. In this case we have a discrete symmetry while in the latter case the fact that the symmetry group can be continuously deformed plays a fundamental role. Perhaps, but this is just speculation, a possible connection can arise if we quantize a $PSL(2, \mathbf{R})$ -invariant theory.

Let me conclude by noticing that modular-invariant hamiltonians or actions can be constructed with a standard procedure. Whenever we find an action $S(\tau)$ depending on a complex coupling constant τ which transforms as $\tau \mapsto \frac{a\tau+b}{c\tau+d}$, and the action itself transforms covariantly as a modular form of weight k see[6], then we can construct a fully modular-invariant action by taking the norm ||S||, obtained by integrating $S\overline{S}$ over the fundamental domain for τ with the help of the Petersson metric. In any such case the above considerations can be applied.

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